Solutions to Problem 1.

a. State space: $\mathcal{M} = \{0, 1, 2, \dots\} \leftarrow$ number of customers in the hair salon

$$\text{Arrival rates,}\quad \lambda_i = \begin{cases} 5 & \text{if } i = 0,1,2,3 \\ 0 & \text{if } i = 4,5,\dots \end{cases} \qquad \text{Service rates,}\quad \mu_i = 6 + (i-1)4 \quad \text{for } i = 1,2,\dots$$

Note that the system will never reach states $5, 6, 7, \dots$, so the service rates $\mu_5, \mu_6, \mu_7, \dots$ are not relevant.

b. First, determine steady-state probabilities:

$$\begin{aligned}
d_0 &= 1 \\
d_1 &= \frac{\lambda_0}{\mu_1} = \frac{5}{6} \\
d_2 &= \frac{\lambda_0\lambda_1}{\mu_1\mu_2} = \frac{5}{6}\left(\frac{5}{10}\right) = \frac{5}{12} \\
d_3 &= \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} = \frac{5}{12}\left(\frac{5}{14}\right) = \frac{25}{168} \\
d_4 &= \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3\mu_4} = \frac{25}{168}\left(\frac{5}{18}\right) = \frac{125}{3024} \\
d_5 &= d_6 = \cdots = 0
\end{aligned}$$

$$\text{Let}\ \ D = \sum_{j=0}^{\infty} d_j \approx 2.44$$

$$\begin{aligned}
\pi_0 &= \frac{d_0}{D} \approx 0.41, & \pi_1 &= \frac{d_1}{D} \approx 0.34, \\
\pi_2 &= \frac{d_2}{D} \approx 0.17, & \pi_3 &= \frac{d_3}{D} \approx 0.06, \\
\pi_4 &= \frac{d_4}{D} \approx 0.02, & \pi_5 &= \pi_6 = \cdots = 0
\end{aligned}$$

Expected number of customers in salon:

$$\ell = \sum_{j=0}^{\infty} j\pi_j \approx 0(0.41) + 1(0.34) + 2(0.17) + 3(0.06) + 4(0.02) = 0.94 \ \ \text{customers}$$

c. Effective arrival rate:

$$\lambda_{\text{eff}} = \sum_{j=0}^{\infty} \lambda_j \pi_j = 4.9 \text{ customers/hour}$$

By Little's law: expected waiting time (includes customers who reneged):

$$w = \frac{\ell}{\lambda_{\text{eff}}} = \frac{0.94}{4.9} \approx 0.192 \text{ hours} = 11.51 \text{ minutes}$$

Solutions to Problem 2.

a. M/M/5 queue with arrival rate $\lambda = 6$ pairs/hour and service rate $\mu = \frac{3}{2}$ pairs/hour (based on an average court use rate of 40 minutes or 2/3 hour).

b.

$$\rho = \frac{\lambda}{s\mu} = \frac{6}{5\left(\frac{3}{2}\right)} = \frac{12}{15} = \frac{4}{5} =$$

$$\pi_0 = \left[\sum_{j=0}^{5} \frac{4^j}{j!} + \frac{5^5\left(\frac{4}{5}\right)^6}{5!\left(\frac{1}{5}\right)}\right]^{-1} = \frac{1}{77} \approx 0.0130$$

c.

$$\pi_5 = \frac{\left(\frac{6}{3/2}\right)^5}{5!}(0.0130) \approx 0.1109$$

$$\ell_q = \frac{\pi_5 \rho}{(1-\rho)^2} \approx \frac{(0.1109)(\frac{4}{5})}{(1-\frac{4}{5})^2} \approx 2.22 \text{ pairs}$$

Note that if you use $\pi_0 \approx 0.01$ instead, you will get very different answers: $\pi_5 \approx 0.0853$ and $\ell_q \approx 1.71$.

d.

$$w_q = \frac{\ell_q}{\lambda} = \frac{2.22}{6} \approx 0.37 \text{ hours} = 22 \text{ minutes}$$

e. Now we model the system as an M/G/5 queue with a service time that is Uniform $[\frac{3}{6}, \frac{5}{6}]$. We can use Whitt's approximation

$$\hat{w_q} = \left(\frac{\epsilon_a + \epsilon_s}{2}\right) w_q$$

together with the result from part d, since the mean service rate remains the same:

$$\varepsilon_a = 1, \text{ since the interarrival times are still exponentially distributed}$$

$$\varepsilon_s = \frac{\text{Var}(X)}{E[X]^2}, \text{ where } X \sim \text{Uniform}\left[\frac{3}{6}, \frac{5}{6}\right]$$

$$= \frac{\frac{1}{12}\left(\frac{5}{6} - \frac{3}{6}\right)^2}{(\frac{4}{6})^2} = \frac{1}{48}$$

Therefore,

$$\hat{w_q} = \frac{1 + \frac{1}{48}}{2} w_q = \frac{49}{96} w_q \approx 0.19 \text{ hours} = 11.4 \text{ minutes}$$

Solutions to Problem 3.

Model as an M/M/∞ queue with arrival rate $\lambda = 12$ and service rate $\mu = 4$.

Let L = a random variable representing the number of call in the system in steady-state. Since this is an M/M/∞ queue, L is a Poisson random variable with parameter $\frac{\lambda}{\mu} = 3$.

We want to find the smallest value of c^* such that $\Pr\{L \leq c^*\} \geq 0.99$, or equivalently:

$$\sum_{j=0}^{c^*} \frac{e^{-3}(3)^j}{j!} \geq 0.99$$

By trial-and-error (starting with $c^* = 1$ and increasing c^* by 1 until the above expression is true), we find that $c^* = 8$.